

# THE HILBERT FUNCTION OF A MAXIMAL COHEN-MACAULAY MODULE

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**ABSTRACT.** We study Hilbert functions of maximal CM modules over CM local rings. We show that if  $A$  is a hypersurface ring with dimension  $d > 0$  then the Hilbert function of  $M$  with respect to  $\mathfrak{m}$  is non-decreasing. If  $A = Q/(f)$  for some regular local ring  $Q$ , we determine a lower bound for  $e_0(M)$  and  $e_1(M)$  and analyze the case when equality holds. When  $A$  is Gorenstein a relation between the second Hilbert coefficient of  $M$ ,  $A$  and  $S^A(M) = (\text{Syz}_1^A(M^*))^*$  is found when  $G(M)$  is CM and  $\text{depth } G(A) \geq d - 1$ . We give bounds for the first Hilbert coefficients of the canonical module of a CM local ring and analyze when equality holds. We also give good bounds on Hilbert coefficients of  $M$  when  $M$  is maximal CM and  $G(M)$  is CM.

## INTRODUCTION

Let  $(A, \mathfrak{m})$  be a  $d$ -dimensional Noetherian local ring and  $M$  a finite  $A$ -module. Let  $G(A) = \bigoplus_{n \geq 0} \mathfrak{m}^n / \mathfrak{m}^{n+1}$  be the associated graded module of  $A$  and  $G(M) = \bigoplus_{n \geq 0} \mathfrak{m}^n M / \mathfrak{m}^{n+1} M$  the associated graded module of  $M$  considered as a  $G(A)$ -module. We set  $\text{depth } G(M) = \text{grade}(\mathcal{M}, G(M))$  where  $\mathcal{M} = \bigoplus_{n \geq 1} \mathfrak{m}^n / \mathfrak{m}^{n+1}$  is the irrelevant maximal ideal of  $G(A)$ . If  $N$  is an  $A$ -module then  $\mu(N)$  denotes its minimal number of generators and  $\lambda(N)$  denotes its length. The Hilbert function of  $M$  (with respect to  $\mathfrak{m}$ ) is the function

$$H(M, n) = \lambda(\mathfrak{m}^n M / \mathfrak{m}^{n+1} M) \text{ for all } n \geq 0$$

In this paper we study Hilbert functions of maximal CM (= MCM) modules. If  $A$  is regular then all MCM modules are free. The next case is that of a hypersurface ring.

**Theorem 1.** *Let  $(A, \mathfrak{m})$  be a hypersurface ring of positive dimension. If  $M$  is a MCM  $A$ -module, then the Hilbert function of  $M$  is non-decreasing.*

This result is a corollary of a more general result (see Theorem 3.3) which also implies that the Hilbert function of a complete intersection of codimension 2 and positive dimension is non-decreasing (see Corollary 3.5). Another application of Theorem 3.3 yields that, if  $(A, \mathfrak{m})$  is equicharacteristic local ring of dimension  $d > 0$ ,  $I$  is an  $\mathfrak{m}$ -primary ideal with  $\mu(I) = d + 1$  and  $M$  is an MCM  $A$ -module, then the Hilbert function of  $M$  with respect to  $I$  is non-decreasing (see Theorem 3.6).

The formal power series

$$H_M(z) = \sum_{n \geq 0} H(M, n) z^n$$

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is called the *Hilbert series* of  $M$ . It is well known that it is of the form

$$H_M(z) = \frac{h_M(z)}{(1-z)^r}, \text{ where } r = \dim M \text{ and } h_M(z) \in \mathbb{Z}[z].$$

We call  $h_M(z)$  the *h-polynomial* of  $M$ . If  $f$  is a polynomial we use  $f^{(i)}$  to denote its  $i$ -th derivative. The integers  $e_i(M) = h_M^{(i)}(1)/i!$  for  $i \geq 0$  are called the *Hilbert coefficients* of  $M$ . The number  $e(M) = e_0(M)$  is the *multiplicity* of  $M$ . Set

$$\chi_i(M) = \sum_{j=0}^i (-1)^{i-j} e_{i-j}(M) + (-1)^{i+1} \mu(M) \text{ for each } i \geq 0.$$

Let  $M$  be a MCM module over a hypersurface ring  $A = Q/(f)$ , where  $(Q, \mathfrak{n})$  is a regular local. If  $0 \rightarrow Q^n \xrightarrow{\phi_M} Q^n \rightarrow M \rightarrow 0$  is a minimal presentation of  $M$  then  $i(M) = \max\{i \mid \text{all entries of } \phi \text{ are in } \mathfrak{n}^i\}$  is an invariant of  $M$ .

**Theorem 2.** *Let  $(Q, \mathfrak{n})$  be a regular local ring,  $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$ ,  $e \geq 2$ ,  $A = Q/(f)$ ,  $M$  a MCM  $A$ -module and  $K = \text{Syz}_1^A(M)$ . Then*

1.  $e(M) \geq \mu(M)i(M)$  and  $e_1(M) \geq \mu(M)\binom{i(M)}{2}$ .
2.  $M$  is a free  $A$ -module if and only if  $i(M) = e$ .
3. If  $i(M) = e - 1$  then  $G(M)$  is CM.
4. The following conditions are equivalent:
  - i.  $e(M) = \mu(M)i(M)$ .
  - ii.  $e_1(M) = \mu(M)\binom{i(M)}{2}$ .
  - iii.  $G(M)$  is CM and  $h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1})$ .

*If these conditions hold and  $M$  is not free, then  $G(K)$  is CM and  $h_K(z) = \mu(M)(1 + z + \dots + z^{e-i(M)-1})$ .*

If  $A$  is complete and  $M$  is a Cohen-Macaulay (= CM)  $A$ -module then there exists a Gorenstein local ring  $R$  such that  $M$  is a MCM  $R$ -module. So it is significant to see how Hilbert functions of MCM modules over Gorenstein rings behave.

When  $A$  is CM,  $M$  is a MCM  $A$ -module and  $N = \text{Syz}_1^A(M)$  then

$$(1) \quad \mu(M)e_1(A) \geq e_1(M) + e_1(N) \quad \& \quad \mu(M)\chi_1(A) \geq \chi_1(M) + \chi_1(N).$$

For the first inequality see [10, 17(3)]. The second follows from [10, 21(1)].

In the theorem below we establish similar inequalities for higher Hilbert coefficients of MCM modules over Gorenstein rings. For every  $A$ -module we set  $M^* = \text{Hom}_A(M, A)$ . Note that if  $M$  is MCM then so is  $M^*$  cf. [2, 3.3.10.d]. Also,  $\text{type}(M) = \dim_k \text{Ext}_A^d(k, M)$  denotes the Cohen-Macaulay type of  $M$ .

**Theorem 3.** *Let  $(A, \mathfrak{m})$  be a Gorenstein local ring. Let  $M$  be a MCM  $A$ -module. Set  $\tau = \text{type}(M)$  and  $S^A(M) = (\text{Syz}_1^A(M^*))^*$ . If  $G(M)$  is CM and  $\text{depth } G(A) \geq d - 1$  then the following hold*

1.  $\tau e_2(A) \geq e_2(M) + e_2(S^A(M))$  and  $\tau \chi_2(A) \geq \chi_2(M) + \chi_2(S^A(M))$ .
2.  $\text{type}(M)e_i(A) \geq e_i(M)$  and  $\text{type}(M)\chi_i(A) \geq \chi_i(M)$  for each  $i \geq 0$ .

Let  $A$  be CM with a canonical module  $\omega_A$ . Set  $\tau = \text{type } A$ . It is well known that  $e_0(\omega_A) = e_0(A)$ . Using [10, Theorem 18] it follows that  $e_1(\omega_A) \leq \tau e_1(A)$  with equality if and only if  $A$  is Gorenstein. Here we give a lower bound on  $e_1(\omega_A)$ .

**Theorem 4.** *Let  $(A, \mathfrak{m})$  be a CM local ring of dimension  $d \geq 1$  and with a canonical module  $\omega_A$ . Set  $\tau = \text{type } A$ . We have*

- (1.)  $\tau^{-1}e_1(A) \leq e_1(\omega_A) \leq \tau e_1(A)$ .
- (2.) (a.)  $e_1(\omega_A) = \tau e_1(A)$  iff  $A$  is Gorenstein.  
 (b.)  $e_1(A) = \tau e_1(\omega_A)$  iff  $A$  is Gorenstein or  $A$  has minimal multiplicity.
- (3.) If  $\dim A = 1$  and  $G(A)$  is CM then  
 $e_i(A) \leq \tau e_i(\omega_A)$  and  $\chi_i(A) \leq \tau \chi_i(\omega_A)$  for each  $i \geq 1$ .

Let  $A = k[[x_1, \dots, x_n]]/\mathfrak{q}$  be CM with  $\mathfrak{q} \subseteq (\mathbf{x})^2$  and  $k$  an infinite field. Let  $1 \leq r \leq d$ . For any two sets of  $r$  sufficiently general  $k$ -linear combinations of  $x_1, \dots, x_n$  say  $y_1, \dots, y_r$  and  $z_1, \dots, z_r$  we show  $H(A/(\mathbf{y}), n) = H(A/(\mathbf{z}), n)$  for each  $n \geq 0$  (see 7.6). We use it to bound Hilbert coefficients of  $M$  if  $G(M)$  is CM (see Theorem 7.8).

Here is an overview of the contents of the paper. In Section 1 we introduce notation and discuss a few preliminary facts that we need. The proof of the Theorems 1 and 3 involves a study of the modules;  $L_t(M) = \bigoplus_{n \geq 0} \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$  for all  $t \geq 0$ . If  $x_1, \dots, x_s$  is a sequence of elements in  $\mathfrak{m}$  then, in section 2, we give  $L_t(M)$  a structure of a graded  $A[X_1, \dots, X_s]$ -module. We prove Theorem 1 in Section 3, Theorem 2 in Section 4 and Theorem 3 in Section 5. We prove Theorem 4 in section 6. In Section 7 we prove Lemma 7.6 and use to prove Theorem 7.8.

## 1. PRELIMINARIES

In this paper all rings are Noetherian and all modules are assumed finite i.e., finitely generated. Let  $(A, \mathfrak{m})$  be a local ring of dimension  $d$  with residue field  $k = A/\mathfrak{m}$ . Let  $M$  be an  $A$ -module. If  $m$  is a non-zero element of  $M$  and if  $j$  is the largest integer such that  $m \in \mathfrak{m}^j M$ , then we let  $m^*$  denote the image of  $m$  in  $\mathfrak{m}^j M / \mathfrak{m}^{j+1} M$ . If  $L$  is a submodule of  $M$ , then  $L^*$  denotes the graded submodule of  $G(M)$  generated by all  $l^*$  with  $l \in L$ . It is well known that  $G(M)/L^* = G(M/L)$ . An element  $x \in \mathfrak{m}$  is said to be *superficial* for  $M$  if there exists an integer  $c > 0$  such that

$$(\mathfrak{m}^n M :_M x) \cap \mathfrak{m}^c M = \mathfrak{m}^{n-1} M \quad \text{for all } n > c.$$

Superficial elements always exist if  $k$  is infinite [11, p. 7]. A sequence  $x_1, x_2, \dots, x_r$  in a local ring  $(A, \mathfrak{m})$  is said to be a *superficial sequence* for  $M$  if  $x_1$  is superficial for  $M$  and  $x_i$  is superficial for  $M/(x_1, \dots, x_{i-1})M$  for  $2 \leq i \leq r$ .

**Remark 1.1.** If the residue field of  $A$  is finite then we resort to the standard trick to replace  $A$  by  $A' = A[X]_S$  and  $M$  by  $M' = M \otimes_A A'$  where  $S = A[X] \setminus \mathfrak{m}A[X]$ . The residue field of  $A'$  is  $k(X)$ , the field of rational functions over  $k$ . Furthermore

$$H(M', n) = H(M, n) \quad \forall n \geq 0 \quad \text{and} \quad \text{depth}_{G(A')} G(M') = \text{depth}_{G(A)} G(M).$$

Clearly  $\text{projdim}_{A'} M' = \text{projdim}_A M$ . If  $A$  is a Gorenstein (hypersurface) ring then  $A'$  is also Gorenstein (hypersurface) ring. If  $A$  has a canonical module  $\omega_A$  then  $A'$  also has a canonical module  $\omega_{A'} \cong \omega_A \otimes A'$ ; cf. [2, Theorem 3.3.14].

Below we collect some basic results needed in the paper. For proofs see [10].

**Remark 1.2.** Let  $(A, \mathfrak{m})$  be a Cohen-Macaulay local ring with  $\dim A = d > 0$ . Let  $M$  be a finite CM  $A$ -module of dimension  $r$ . Let  $x_1, \dots, x_s$  be a superficial sequence in  $M$  with  $s \leq r$  and set  $J = (x_1, \dots, x_s)$ . The local ring  $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$  and  $B$ -module  $N = M/JM$  satisfy:

- 1.  $x_1, \dots, x_s$  is a  $M$ -regular sequence in  $A$ .
- 2.  $N$  is a CM  $B$ -module.

3.  $e_i(N) = e_i(M)$  for  $i = 0, 1, \dots, r - s$ .
4. When  $s = 1$ , set  $x = x_1$  and  $b_n(x, M) = \lambda(\mathfrak{m}^{n+1}M :_M x)/\mathfrak{m}^n M$ . We have:
  - a.  $b_0(x, M) = 0$  and  $b_n(x, M) = 0$  for all  $n \gg 0$ .
  - b.  $H(M, n) = \sum_{i=0}^n H(N, i) - b_n(x, M)$
  - c.  $e_r(M) = e_r(N) - (-1)^r \sum_{i \geq 0} b_i(x, M)$ .
  - d.  $x^*$  is  $G(M)$ -regular if and only if  $b_n(x, M) = 0$  for all  $n \geq 0$ .
5. a.  $\text{depth } G(M) \geq s$  if and only if  $x_1^*, \dots, x_s^*$  is a  $G(M)$  regular sequence.  
 b. (*Sally descent*)  $\text{depth } G(M) \geq s + 1$  if and only if  $\text{depth } G(N) \geq 1$ .
6. If  $\dim M = 1$  then set  $\rho_n(M) = \lambda(\mathfrak{m}^{n+1}M/x\mathfrak{m}^n M)$ . We have
  - a.  $H(M, n) = e(M) - \rho_n(M)$ .
  - b.  $e_i(M) = \sum_{j \geq i-1} \binom{j}{i-1} \rho_j(M) \geq 0$  for all  $i \geq 1$ .
7. If  $x_1, \dots, x_s$  is also  $A$ -regular then  $\text{Syz}_1^B(N) \cong \text{Syz}_1^A(M)/J \text{Syz}_1^A(M)$
8.  $\text{depth } G(M) \geq s$  if and only if  $h_M(z) = h_N(z)$ .
9.  $M$  has minimal multiplicity if and only if  $\chi_1(M) = 0$ .

**Remark 1.3.** If  $\phi : (A, \mathfrak{m}) \longrightarrow (B, \mathfrak{n})$  is a surjective map of local rings and if  $M$  is a finite  $B$ -module then  $\mathfrak{m}^n M = \mathfrak{n}^n M$  for all  $n \geq 0$ . Therefore  $G_{\mathfrak{m}}(M) = G_{\mathfrak{n}}(M)$ . The notation  $G(M)$  will be used to denote this without any reference to the ring. Also note that  $\text{depth}_{G_{\mathfrak{m}}(A)} G(M) = \text{depth}_{G_{\mathfrak{n}}(B)} G(M)$ .

**1.4.** Recall that the function  $n \mapsto \lambda(M/\mathfrak{m}^{n+1}M)$  is called the *Hilbert-Samuel* function. Let  $p_M(z)$  be the *Hilbert-Samuel polynomial*. The following number

$$(2) \quad \text{post}(M) = \min\{n \mid p_M(i) = \lambda(M/\mathfrak{m}^{i+1}M) \text{ for all } i \geq n\}$$

is called the *postulation number* of  $M$  (with respect to  $\mathfrak{m}$ ).

**1.5.** If  $f(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{Z}[z]$  then for  $i \geq 0$  set  $e_i(f) = f^{(i)}(1)/i! = \sum_{k \geq i} \binom{k}{i} a_k$  and set

$$\chi_i(f) = \sum_{j=0}^i (-1)^{i-j} e_{i-j}(f) + (-1)^{i+1} f(0) = \sum_{k \geq i+1} \binom{k-1}{i} a_k.$$

It follows that if  $a_i \geq 0$  for all  $i \geq 0$  then  $e_i(f) \geq 0$  and  $\chi_i(f) \geq 0$  for all  $i \geq 0$ . The following Lemma can be easily proved.

**Lemma 1.6.** *If  $g(z), p(z), q(z)$  and  $r(z)$  are polynomials with integer coefficients that satisfy the equation  $(1-z)g(z) = p(z) - q(z) + r(z)$  then*

- (i)  $e_0(q) = e_0(p) + e_0(r)$ .
- (ii)  $e_i(q) = e_i(p) + e_i(r) + e_{i-1}(g)$  for  $i \geq 1$ .
- (iii)  $\chi_0(q) = \chi_0(p) + \chi_0(r) + g(0)$ .
- (iv)  $\chi_i(q) = \chi_i(p) + \chi_i(r) + \chi_{i-1}(g)$  for  $i \geq 1$ .
- (v) *If all the coefficients of  $g$  are non-negative then for  $i \geq 0$  we have*  
 $e_i(q) \geq e_i(p) + e_i(r)$  and  $\chi_i(q) \geq \chi_i(p) + \chi_i(r)$ . □

## 2. BASIC CONSTRUCTION

**Remark 2.1.** For each  $n \geq 0$  and  $t \geq 0$  set  $L_t(M)_n = \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$ . For  $t \geq 0$  let  $L_t(M) = \bigoplus_{n \geq 0} L_t(M)_n$ . If  $x_1, \dots, x_s$  is a sequence of elements in  $\mathfrak{m}$ , then we give  $L_t(M)$  a structure of a graded  $A[X_1, \dots, X_s]$ -module as follows:

For  $i = 1, \dots, s$  let  $\xi_i : A/\mathfrak{m}^n \rightarrow A/\mathfrak{m}^{n+1}$  be the maps given by  $\xi_i(a + \mathfrak{m}^n) = x_i a + \mathfrak{m}^{n+1}$ . These homomorphisms induces homomorphisms

$$\text{Tor}_t^A(\xi_i, M) : \text{Tor}_t^A(A/\mathfrak{m}^n, M) \longrightarrow \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$$

Thus, for  $i = 1, \dots, s$  and each  $t$  we obtain homogeneous maps of degree 1:

$$X_i : L_t(M) \longrightarrow L_t(M).$$

For  $i, j = 1, \dots, s$  the equalities  $\xi_i \xi_j = \xi_j \xi_i$  yields equalities  $X_i X_j = X_j X_i$ . So  $L_t(M)$  is a graded  $A[X_1, \dots, X_s]$ -module for each  $t \geq 0$ .

**Proposition 2.2.** *Let  $M, F$ , and  $K$  be finite  $A$ -modules and let  $x_1, \dots, x_s$  be a sequence of elements in  $\mathfrak{m}$ . If  $L_t(M)$ ,  $L_t(F)$  and  $L_t(K)$  are given the  $A[X_1, \dots, X_s]$ -module structure described in Remark 2.1 then*

1. *Every exact sequence of  $A$ -modules  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$  induces a long exact sequence of graded  $A[X_1, \dots, X_s]$ -modules*  

$$\cdots \rightarrow L_{t+1}(M) \rightarrow L_t(K) \rightarrow L_t(F) \rightarrow L_t(M) \rightarrow \cdots \rightarrow L_0(M) \rightarrow 0.$$
2. *For  $i = 1, \dots, s$  there is an equality*

$$\ker (L_0(M)_{n-1} \xrightarrow{X_i} L_0(M)_n) = \frac{\mathfrak{m}^{n+1}M :_M x_i}{\mathfrak{m}^n M}$$

3. *If  $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$  is such that  $x_i^*$  is  $G(M)$ -regular then  $X_i$  is  $L_0(M)$ -regular.*
4. *If  $F$  is free  $A$ -module and  $x_i$  is  $K$ -superficial for some  $i$  then*
  - (a)  $\ker (L_1(M) \xrightarrow{X_i} L_1(M))_n = 0$  for  $n \gg 0$
  - (b) *If  $x_i^*$  is  $G(K)$ -regular then  $X_i$  is  $L_1(M)$ -regular.*

*Proof.* To prove part 1, set  $S = A[X_1, \dots, X_s]$ ,

$$\beta_{t,n} = \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, \beta) : \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, K) \longrightarrow \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, F)$$

$$\alpha_{t,n} = \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, \alpha) : \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, F) \longrightarrow \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, M)$$

and consider the connecting homomorphisms

$$\delta_{t+1,n} : \text{Tor}_{t+1}^A(A/\mathfrak{m}^{n+1}, M) \longrightarrow \text{Tor}_t^A(A/\mathfrak{m}^{n+1}, K).$$

By a well known theorem in Homological algebra if  $\mathbf{X}$  is a free resolution of  $K$  and  $\mathbf{Z}$  is a free resolution of  $M$  then there exists a free resolution  $\mathbf{Y}$  of  $F$  and an exact sequence of complexes of free  $A$ -modules  $0 \rightarrow \mathbf{X} \rightarrow \mathbf{Y} \rightarrow \mathbf{Z} \rightarrow 0$  whose homology sequence is the given exact sequence  $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ . This yields for each  $n$  a commuting diagram of complexes with exact rows ;

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{X} \otimes A/\mathfrak{m}^n & \longrightarrow & \mathbf{Y} \otimes A/\mathfrak{m}^n & \longrightarrow & \mathbf{Z} \otimes A/\mathfrak{m}^n \longrightarrow 0 \\ & & \downarrow \xi_i & & \downarrow \xi_i & & \downarrow \xi_i \\ 0 & \longrightarrow & \mathbf{X} \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \mathbf{Y} \otimes A/\mathfrak{m}^{n+1} & \longrightarrow & \mathbf{Z} \otimes A/\mathfrak{m}^{n+1} \longrightarrow 0 \end{array}$$

In homology it induces the following commutative diagram :

$$\begin{array}{ccccccc} \cdots & \longrightarrow & L_{t+1}(M)_{n-1} & \xrightarrow{\delta_{t+1,n-1}} & L_t(K)_{n-1} & \xrightarrow{\beta_{t,n-1}} & L_t(F)_{n-1} \xrightarrow{\alpha_{t,n-1}} \cdots \\ & & \downarrow X_i & & \downarrow X_i & & \downarrow X_i \\ \cdots & \longrightarrow & L_{t+1}(M)_n & \xrightarrow{\delta_{t+1,n}} & L_t(K)_n & \xrightarrow{\beta_{t,n}} & L_t(F)_n \xrightarrow{\alpha_{t,n}} \cdots \end{array}$$

This proves the desired assertion.

**Remark 2.3.** We will use the exact diagram above often. So when there is a reference to this remark, I mean to refer the commuting diagram above.

The second part is clear from the definition of the action  $X_i$ . Part 3. follows from 2. If  $F$  is free, then  $L_1(F) = 0$ , so 1. gives an exact sequence of  $S$ -modules  $0 \rightarrow L_1(M) \rightarrow L_0(K)$ . Together with 2. and 3. this yields the assertions in 4.  $\square$

**Remark 2.4.** If  $(A, \mathfrak{m})$  is CM of dimension  $d$  and  $M$  is maximal non-free CM then by [10, Remark 23] there is an equality

$$(3) \quad \sum_{n \geq 0} \lambda(\text{Tor}_1^A(M, A/\mathfrak{m}^{n+1})) z^n = \frac{l_M(z)}{(1-z)^d} \quad \text{here } l_M(z) \in \mathbb{Z}[z] \text{ and } l_M(1) \neq 0$$

$$(4) \quad (1-z)l_M(z) = h_{\text{Syz}_1^A(M)}(z) - \mu(M)h_A(z) + h_M(z).$$

We study the case when  $\dim A = 0$ .

**Lemma 2.5.** *If  $\dim A = 0$  and  $M$  is any finite  $A$ -module then for all  $i \geq 0$*

1.  $\mu(M)e_i(A) \geq e_i(M) + e_i(\text{Syz}_1^A(M))$  and  $\mu(M)\chi_i(A) \geq \chi_i(M) + \chi_i(\text{Syz}_1^A(M))$ .
2.  $\mu(M)e_i(A) \geq e_i(M)$  and  $\mu(M)\chi_i(A) \geq \chi_i(M)$ .

*Proof.* Note that when  $\dim A = 0$  we get that  $l_M(z)$  has non-negative coefficients. Using (4) and Lemma 1.6.v we get 1. and 2. The assertion 3. follows from 1. and 2. since  $N = \text{Syz}_1^A(M)$  has dimension zero and so  $h_N(z)$  has non-negative coefficients. Therefore  $e_i(N)$  and  $\chi_i(N)$  are non-negative for  $i \geq 0$  (see (1.5)).  $\square$

**2.6.** It follows from Lemma 2.5.3 that if  $G(A)$  and  $G(M)$  is CM then  $e_i(A)\mu(M) \geq e_i(M)$  for all  $i \geq 0$ .

**Remark 2.7.** In view of the Remark 2.4 and (2.6) it is quite important to understand  $L^1(M) = \bigoplus_{n \geq 0} \text{Tor}_1^A(M, A/\mathfrak{m}^{n+1})$  when  $M$  is MCM. In the next Lemma we answer the question when  $\dim M = 1$ .

**Lemma 2.8.** *Let  $(A, \mathfrak{m})$  be a Cohen-Macaulay local ring of dimension one, let  $M$  be a non-free maximal Cohen-Macaulay  $A$ -modules and let*

$$0 \longrightarrow E \longrightarrow F \longrightarrow M \longrightarrow 0$$

*be an exact sequence with  $F$  a finite free  $A$ -module. Let  $x$  be  $A \oplus M \oplus E$ -superficial. If  $L_1(M)$  is given the  $A[X]$ -module structure described in Remark 2.1 then we have*

1. *There is an  $\mathfrak{m}$ -primary ideal  $\mathfrak{q}$  such that  $\mathfrak{q}A[X]L_1(M) = 0$ . Furthermore  $L_1(M)$  is a Noetherian  $A/\mathfrak{q}[X]$ -module of dimension one.*
2.  $(1-z)l_M(z) = h_E(z) - h_F(z) + h_M(z)$ .
3. *If  $G(E)$  is CM then  $X$  is  $L_1(M)$ -regular. Furthermore*

$$e_i(F) \geq e_i(M) + e_i(E) \quad \text{and} \quad \chi_i(F) \geq \chi_i(M) + \chi_i(E) \quad \text{for all } i \geq 0.$$

*Proof.* 1. Since  $\dim A = 1$  and  $M$  is non-free, it follows from Lemma 2.8 that  $\lambda(L_1(M)_n)$  is a non-zero constant for large  $n$ . Since  $X: L_1(M)_n \rightarrow L_1(M)_{n+1}$  is injective for large  $n$  and since  $\lambda(L_1(M)_n)$  is constant for large  $n$ , it follows that  $L_1(M)_{n+1} = XL_1(M)_n$  for large  $n$ , say for all  $n \geq s$ . For  $n \geq 0$  set  $\mathfrak{q}_n = \text{ann}_A L_1(M)_n$ . Note that  $\mathfrak{q}_n$  is  $\mathfrak{m}$ -primary for all  $n$ . Since the map  $X: L_1(M)_n \rightarrow L_1(M)_{n+1}$  is bijective for all  $n \geq s$  we have  $\mathfrak{q}_n = \mathfrak{q}_s$  for each  $n \geq s$ . Set  $\mathfrak{q} = \bigcap_{n=0}^s \mathfrak{q}_n$ . Clearly  $\mathfrak{q}L_1(M)_n = 0$  for each  $n \geq 0$ . Thus  $L_1(M)$  is an  $A/\mathfrak{q}[X]$  module. For each  $i = 0, 1, \dots, s$  choose a finite set  $\mathcal{P}_i$  of generators of  $L_1(M)_i$  as an  $A$ -module. It is easy to see that  $\bigcup_{i=0}^s \mathcal{P}_i$  generates  $L_1(M)$  over  $A[X]$ . Since  $\lambda(L_1(M)/XL_1(M)) < \infty$  and  $\lambda(L_1(M)) = \infty$  it follows that  $\dim L_1(M) = 1$ .

2. By Schanuel's lemma, [9, p. 158] we have  $F \oplus \text{Syz}_1^A(M) \cong E \oplus A^{\mu(M)}$ . Therefore  $(1-z)l_M(z) = h_{\text{Syz}_1^A(M)}(z) - \mu(M)h_A(z) + h_M(z) = h_E(z) - h_F(z) + h_M(z)$ .

3. It is clear from 1. and Proposition 2.2.3 that  $X$  is  $L_1(M)$ -regular. It also follows from 1. that  $l_M(z)$  is the  $h$ -polynomial of  $L_1(M)$  considered as an  $A[X]$ -module. Set  $e_i^T(M) = l_M^{(i)}(l)/i!$ . Since  $X$  is  $L_1(M)$ -regular and  $\dim L_1(M) = 1$  we have that  $l_M(z)$  is the Hilbert series of  $L_1(M)/XL_1(M)$ . Thus all the coefficients of  $l_M(z)$  is non-negative. Using 2. and Lemma 1.6.v we get the desired inequalities.  $\square$

### 3. MONOTONICITY

The following remark will be used often.

**Remark 3.1.** Let  $f(z) = \sum_{n \geq 0} a_n z^n$  be a formal power series with non-negative coefficients. If the power series  $g(z) = \sum_{n \geq 0} b_n z^n$  satisfies  $g(z) = f(z)/(1-z)$ , then  $b_n = \sum_{i=0}^n a_i$ , and so the sequence  $\{b_n\}$  is nondecreasing.

The next proposition yields an easy criterion for monotonicity.

**Proposition 3.2.** *Let  $M$  be an  $A$ -module. Set  $k = A/\mathfrak{m}$ . If  $\text{depth } G(M) \geq 1$  then the Hilbert function of  $M$  is non-decreasing.*

*Proof.* Using Remark 1.1 we may assume that  $k$  is infinite. Thus there exists  $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ , such that  $x^*$  is  $G(M)$ -regular. It follows that the Hilbert function of  $M$  is non-decreasing.  $\square$

We deduce Theorem 1 from the following result.

**Theorem 3.3.** *Let  $(Q, \mathfrak{m})$  be a local ring with  $\text{depth } G(Q) \geq 2$  and let  $M$  be a  $Q$ -module. If  $\text{projdim}_Q M \leq 1$  then the Hilbert function of  $M$  is non-decreasing.*

*Proof.* It is sufficient to consider the case when the residue field of  $Q$  is infinite (see Remark 1.1). Since  $\text{projdim}_Q M \leq 1$  we have a presentation of  $M$

$$(5) \quad 0 \longrightarrow Q^n \longrightarrow Q^m \longrightarrow M \longrightarrow 0 \quad \text{with } 0 \leq n \leq m.$$

Let  $x, y$  be elements in  $\mathfrak{m} \setminus \mathfrak{m}^2$  such that  $x^*, y^*$  is a  $G(Q)$ -regular sequence. Let  $L_0(Q)$ ,  $L_0(M)$  and  $L_1(M)$  be the  $Q[X, Y]$ -modules described in Remark 2.1.

By Proposition 2.2.3 we get  $X$  is  $L_0(Q)$ -regular. Set  $B = Q/(x)$  and notice

$$\frac{L_0(Q)}{XL_0(Q)} = \bigoplus_{n \geq 0} \frac{Q}{(x, \mathfrak{m}^{n+1})} = L_0(B).$$

Since  $G(B) = G(Q)/x^*G(Q)$  we see that  $y^*$  is  $G(B)$ -regular. Proposition 2.2.3 shows that  $Y$  is  $L_0(B)$ -regular. Thus  $X, Y$  is a  $L_0(Q)$ -regular sequence.

Using the exact sequence (5) and Proposition 2.2.1, we obtain an exact sequence of graded  $Q[X, Y]$  modules

$$(6) \quad 0 \longrightarrow L_1(M) \longrightarrow L_0(Q)^n \xrightarrow{\phi} L_0(Q)^m \longrightarrow L_0(M) \longrightarrow 0.$$

Set  $K = \text{image } \phi$ . Since  $X$  is  $L_0(Q)$  regular we see that it is both  $K$  and  $L_1(M)$ -regular. So the exact sequence  $0 \rightarrow L_1(M) \rightarrow L_0(Q)^n \rightarrow K \rightarrow 0$  yields the exact sequence

$$0 \longrightarrow \frac{L_1(M)}{XL_1(M)} \longrightarrow \frac{L_0(Q)^n}{XL_0(Q)^n} \longrightarrow \frac{K}{XK} \longrightarrow 0$$

Since  $Y$  is  $L_0(Q)^n/XL_0(Q)^n$  regular it follows that  $Y$  is  $L_1(M)/XL_1(M)$ -regular. Thus  $X, Y$  is an  $L_1(M)$ -regular sequence.

The regularity of  $X, Y$  implies equalities

$$\sum_{i \geq 0} \lambda(L_0(Q)_i) z^i = \frac{u(z)}{(1-z)^2} \quad \text{where} \quad u(z) = \sum_{i \geq 0} \lambda \left( \frac{L_0(Q)_i}{(X, Y)L_0(Q)_{i-1}} \right) z^i.$$

$$\sum_{i \geq 0} \lambda(L_1(M)_i) z^i = \frac{v(z)}{(1-z)^2} \quad \text{where} \quad v(z) = \sum_{i \geq 0} \lambda \left( \frac{L_1(M)_i}{(X, Y)L_1(M)_{i-1}} \right) z^i.$$

Using the exact sequence (6) we get

$$\sum_{i \geq 0} \lambda(L_0(M)_i) z^i = (m-n) \frac{u(z)}{(1-z)^2} + \frac{v(z)}{(1-z)^2}$$

The equality  $H_M(z) = (1-z) \sum_{i \geq 0} \lambda(L_0(M)_i) z^i$  yields

$$H_M(z) = (m-n)u(z)/(1-z) + v(z)/(1-z).$$

Now Remark 3.1 shows that the Hilbert function of  $M$  is non-decreasing.  $\square$

We obtain Theorem 1 as a corollary to the previous theorem.

*Proof of Theorem 1.* We may assume that  $A$  is complete and so  $A \cong Q/(f)$  for some regular local ring  $(Q, \mathfrak{n})$  and  $f \in \mathfrak{n}^2$ . Then  $\text{depth } M = \dim Q - 1$  and  $\text{projdim}_Q M = 1$ . Using Theorem 3.3 it follows that the Hilbert function of  $M$  is non-decreasing.  $\square$

Since the Hilbert function is increasing if  $\text{depth } G(M) > 0$ , we construct a MCM module  $M$  over a hypersurface ring  $A$  such that  $\text{depth } G(M) = 0$ .

**Example 3.4.** Set  $Q = k[[x, y]]$  and  $\mathfrak{n} = (x, y)$ . Define  $M$  by the exact sequence

$$0 \longrightarrow Q^2 \xrightarrow{\phi} Q^2 \longrightarrow M \longrightarrow 0 \quad \text{where}$$

$$\phi = \begin{pmatrix} x & y \\ -y^2 & 0 \end{pmatrix}$$

Set  $(A, \mathfrak{m}) = (Q/(y^3), \mathfrak{n}/(y^3))$ . Note  $y^3 = \det(\phi)$  annihilates  $M$ . So  $M$  is a MCM  $A$ -module. Set  $K = \text{Syz}_1^A(M)$ . Note that  $G(Q) = k[x^*, y^*]$ . Since  $y^3 M = 0$ , we have that if  $P \in \text{Ass}_{G(Q)}(G(M))$  then  $P \supseteq (y^*)$ . So we get that  $x^* \notin P$  if  $P$  is a relevant associated prime of  $G(M)$ . Therefore  $x^*$  is an  $M$ -superficial element. We show  $\text{depth } G(M) = 0$ . Otherwise by 1.2.5.a we get that  $x^*$  is  $G(M)$ -regular. However if  $m_1, m_2$  are the generators of  $M$  then  $xm_1 = y^2 m_2 \in \mathfrak{n}^2 M$  and this implies  $m_1 \in (\mathfrak{n}^2 M :_M x) = \mathfrak{n} M$ , which is a contradiction.

The next corollary partly overlaps with a result of Elias [5]: all equicharacteristic CM rings of dimension 1 and embedding dimension 3 have non-decreasing Hilbert functions. See [13, p. 337] for an example of a complete intersection ring  $(A, \mathfrak{m})$  of dimension 1 and codimension 2 such that  $\text{depth } G(A) = 0$ .

**Corollary 3.5.** *If  $(A, \mathfrak{m})$  be a complete intersection of positive dimension and codimension 2 then the Hilbert function of  $A$  is non-decreasing.*

*Proof.* We may assume that  $A$  is complete and hence  $A = Q/(f, g)$  for a regular sequence  $f, g$  in a regular local ring  $(Q, \mathfrak{q})$ . Set  $(R, \mathfrak{n}) = (Q/(f), \mathfrak{q}/(f))$ . Then  $G(R)$  is Cohen-Macaulay,  $\dim A = \dim R - 1$  and  $\text{projdim}_R A = 1$ . Therefore by Theorem 3.3 we get that the Hilbert function of  $A$  is non-decreasing.  $\square$

Another application of Theorem 3.3 yields the following:

**Theorem 3.6.** *Let  $(A, \mathfrak{m})$  be a Noetherian equicharacteristic local ring of dimension  $d > 0$  and let  $M$  be a MCM  $A$ -module. Let  $I$  be an  $\mathfrak{m}$ -primary ideal in  $A$  with  $\mu(I) = d + 1$ . Then the Hilbert function of  $M$  with respect to  $I$  is non-decreasing.*

*Proof.* Without any loss of generality we may assume that  $A$  is complete. Let  $I = (x_1, \dots, x_{d+1})$ . Since  $A$  is complete and equicharacteristic it contains a subfield  $k \cong A/\mathfrak{m}$ . Set  $R = k[[T_1, \dots, T_{d+1}]]$  and let  $\mathfrak{n}$  be its unique maximal ideal. Consider the local homomorphism  $\phi : R \rightarrow A$  defined by  $\phi(T_i) = x_i$ . Then  $A$  becomes an  $R$ -module via  $\phi$ . Since  $A/\mathfrak{n}A = A/I$  has finite length we get  $M$  is a finite  $R$ -module. It can be easily checked that  $M$  is a CM  $R$ -module of dimension  $d$ .

Since  $R$  is regular,  $\text{projdim}_R M$  is finite. So  $\text{projdim}_R M = \text{depth } R - \text{depth } M = 1$ . Therefore by Theorem 3.3 it follows that  $H_{\mathfrak{n}}(M, j)$  is non-decreasing. Note that  $\mathfrak{n}^j M = I^j M$  for each  $j \geq 1$  and so  $H_{\mathfrak{n}}(M, j) = \lambda(I^j M / I^{j+1} M)$  for each  $j \geq 0$ . This establishes the assertion of the theorem.  $\square$

#### 4. HILBERT COEFFICIENTS

**4.1.** In this section  $\varepsilon_s$  denotes the  $s \times s$  identity matrix. Let  $(Q, \mathfrak{n})$  be a regular local ring,  $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$ ,  $e \geq 2$ ,  $A = Q/(f)$ ,  $M$  a MCM  $A$ -module and  $K = \text{Syz}_1^A(M)$ .

By a *matrix-factorization* of  $f$  we mean a pair  $(\phi, \psi)$  of square-matrices with elements in  $Q$  such that

$$\phi\psi = \psi\phi = f\varepsilon.$$

If  $M$  is an  $A$ -module then  $\text{projdim}_Q M = 1$ . Also a presentation of  $M$

$$0 \longrightarrow Q^n \xrightarrow{\phi} Q^n \longrightarrow M \longrightarrow 0$$

yields a matrix factorization of  $f$ . See [4, p. 53] for details.

In the sequel  $(\phi_M, \psi_M)$  will denote a matrix factorization of  $f$  such that

$$0 \longrightarrow Q^n \xrightarrow{\phi_M} Q^n \longrightarrow M \longrightarrow 0$$

is a minimal presentation of  $M$ . Note that

$$0 \longrightarrow Q^n \xrightarrow{\psi_M} Q^n \longrightarrow \text{Syz}_1^A(M) \longrightarrow 0$$

is a not-necessarily minimal presentation of  $\text{Syz}_1^A(M)$ .

If  $\phi : Q^n \longrightarrow Q^m$  is a linear map then we set

$$i_\phi = \max\{i \mid \text{all entries of } \phi \text{ are in } \mathfrak{n}^i\}.$$

If  $M$  has minimal presentations:  $0 \rightarrow Q^n \xrightarrow{\phi} Q^n \rightarrow M \rightarrow 0$  and

$0 \rightarrow Q^n \xrightarrow{\phi'} Q^n \rightarrow M \rightarrow 0$ , then it is well known that  $i_\phi = i_{\phi'}$  and  $\det(\phi) = u \det(\phi')$  with  $u$  a unit. We set  $i(M) = i_\phi$  and  $\det(M) = (\det(\phi))$ . For  $g \in Q$ ,  $g \neq 0$ , set  $v_Q(g) = \max\{i \mid g \in \mathfrak{n}^i\}$ . For convenience set  $v_Q(0) = \infty$ . Note that  $e(Q/(g)) = v_Q(g)$  for any  $g \neq 0$ . We first consider the case when  $\dim A = 0$ .

**Remark 4.2.** Let  $(Q, \mathfrak{n})$  be a DVR,  $v_Q(f) = e$ ,  $A = Q/(f)$  and  $M$  a finite  $A$ -module. If  $\mathfrak{n} = (y)$  then  $f = uy^e$ , where  $u$  is a unit. Therefore as an  $Q$ -module

$$M \cong \bigoplus_{i=1}^{\mu(M)} Q/(y^{a_i}) \quad \text{with } 1 \leq a_1 \leq \dots \leq a_{\mu(M)} \leq e.$$

This yields a minimal presentation of  $M$ :

$$0 \rightarrow Q^n \xrightarrow{\psi} Q^n \rightarrow M \rightarrow 0 \text{ where } \psi_{ij} = \delta_{ij}y^{a_i}.$$

This yields

- (1)  $i(M) = a_1$ .
- (2)  $h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1}) + \text{higher powers of } z$
- (3)  $h_0(M) \geq h_1(M) \geq \dots \geq h_s(M)$ .
- (4)  $M$  is free if and only if  $i(M) = e$ .
- (5) As an  $Q$ -module  $K \cong \bigoplus_{i=1}^{\mu(M)} Q/(y^{e-a_i})$ .
- (6)  $e(M) \geq \mu(M)i(M)$  and  $e_1(M) \geq \mu(M)\binom{i(M)}{2}$ .
- (7)  $v_Q(\det \phi) \geq i(M)\mu(M)$  with equality iff  $e(M) = i(M)\mu(M)$ .

We note an immediate corollary to assertion 3. in the previous remark.

**Corollary 4.3.** Let  $(A, \mathfrak{m})$  be a hypersurface ring of dimension  $d$ . Let  $M$  be a MCM  $A$ -module such that  $G(M)$  is CM. If  $h_M(z) = h_0(M) + h_1(M)z + \dots + h_s(M)z^s$  is the  $h$ -polynomial of  $M$  then  $h_0(M) \geq h_1(M) \geq \dots \geq h_s(M)$ .

*Proof.* We may assume that  $A$  is complete with infinite residue field, hence  $A = Q/(f)$  for some regular local ring  $(Q, \mathfrak{n})$ . Consider  $M$  as a  $Q$ -module. Let  $x_1, \dots, x_d$  be a  $Q \oplus M$ -superficial sequence. Set  $J = (x_1, \dots, x_d)$ ,  $(R, \mathfrak{q}) = (Q/J, \mathfrak{n}/J)$ ,  $B = A/J$  and  $N = M/JM$ . Note that  $R$  is a DVR. Since  $G(M)$  is CM we also have  $h_M(z) = h_N(z)$  and so the result follows from Remark 4.2(3).  $\square$

To use the other assertions in Remark 4.2 we need the following definitions. The notion of superficial sequence is extremely useful in the study of Hilbert functions. We need to generalize it to deal also with a presentation of a module.

**Definition 4.4.** Let  $(Q, \mathfrak{n})$  be a regular local ring,  $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$ ,  $e \geq 2$ ,  $A = Q/(f)$  and  $M$  a MCM  $A$ -module. Let  $0 \rightarrow Q^n \xrightarrow{\phi} Q^n \rightarrow M \rightarrow 0$  be a minimal presentation of  $M$ . We say that  $x \in \mathfrak{n}$  is  $\phi$ -superficial if

- (1)  $x$  is  $(Q \oplus M \oplus A)$ -superficial.
- (2) If  $\phi = (\phi_{ij})$  then  $v_Q(\phi_{ij}) = v_{Q/xQ}(\overline{\phi_{ij}})$
- (3)  $v_Q(\det(\phi)) = v_{Q/xQ}(\det(\overline{\phi}))$ .

Since  $e(Q/(g)) = v_Q(g)$  for any  $g \neq 0$  it follows that if  $x$  is  $Q \oplus M \oplus A \oplus (\bigoplus_{ij} Q/(\phi_{ij})) \oplus Q/\det(\phi)$ -superficial then it is  $\phi$ -superficial. So  $\phi$ -superficial elements exist if the residue field of  $Q$  is infinite.

If  $x$  is  $\phi$ -superficial, then clearly  $i(M) = i(M/xM)$ . Also note that  $Q/xQ$  is regular and we have an exact sequence

$$0 \longrightarrow \left( \frac{Q}{xQ} \right)^n \xrightarrow{\phi \otimes_Q Q/xQ} \left( \frac{Q}{xQ} \right)^n \longrightarrow \frac{M}{xM} \longrightarrow 0$$

This enables the following definition:

**Definition 4.5.** Let  $(Q, \mathfrak{n})$  be a regular local ring,  $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$ ,  $e \geq 2$ ,  $A = Q/(f)$  and  $M$  a MCM  $A$ -module. Let  $0 \rightarrow Q^n \xrightarrow{\phi} Q^n \rightarrow M \rightarrow 0$  be a minimal presentation of  $M$ . We say that  $x_1, \dots, x_r$  is a  $\phi$ -superficial sequence if  $\overline{x_i}$  is  $(\phi \otimes_Q Q/(x_1, \dots, x_{i-1}))$ -superficial for  $i = 1, \dots, r$ .

**Notation:** Let  $M$  be an  $A$ -module. If  $x$  is  $A \oplus M$  superficial (or more generally it is superficial with respect to an injective map  $\theta : Q^n \rightarrow Q^n$ ) then set  $(B, \mathfrak{n}) = (A/(x), \mathfrak{m}/(x))$  and  $N = M/xM$ .

We need a few preliminaries before we prove Theorem 2.

**Lemma 4.6.** Let  $(A, \mathfrak{m})$  be a CM local ring of dimension  $d > 0$  with infinite residue field. Let  $M$  be a CM  $A$ -module of dimension 1 with a presentation  $G \xrightarrow{\phi} F \rightarrow M \rightarrow 0$  such that all entries in  $\phi$  are in  $\mathfrak{m}^l$ . If  $\text{depth } G(A) \geq 1$  and  $x$  is a  $A \oplus M$ -superficial element then

1.  $(\mathfrak{m}^{i+1}M :_M x) = \mathfrak{m}^i M$  for  $i = 0, \dots, l-1$ .
2. Furthermore if  $\mathfrak{m}^l M \subseteq xM$  then  $\text{depth } G(M) \geq 1$ .

*Proof.* Set  $b_i(M) = \lambda((\mathfrak{m}^{i+1}M :_M x)/\mathfrak{m}^i M)$ . Since all the entries of  $\phi$  are in  $\mathfrak{m}^l$  we have that  $\phi_{0,j-1} = \phi \otimes A/\mathfrak{m}^j = 0$  for  $j = 1, \dots, l$ .

1. Note that  $b_0(K) = 0$  for any  $A$ -module  $K$ . Fix  $i$  with  $1 \leq i \leq l-1$ . Let  $p \in (\mathfrak{m}^{i+1}M :_M x)$ . Let  $u$  be the pre-image of  $p$  in  $F$ . Using the commutative diagram 2.3 and since  $\phi_{0,i} = 0$  and  $\phi_{0,i+1} = 0$  we obtain that  $xu \in \mathfrak{m}^{i+1}F$ . Since  $x$  is  $A$ -superficial and  $\text{depth } G(A) \geq 1$  we get by 1.2.5.a that  $x^*$  is also  $G(A)$ -regular. Therefore  $u \in \mathfrak{m}^i F$  and so  $p \in \mathfrak{m}^i M$ . Thus  $b_i(M) = 0$ .

2. Since  $\mathfrak{m}^l M \subseteq xM$  we get that  $\mathfrak{n}^l N = 0$  and so  $\sum_{i=0}^{l-1} H(N, i) = e(N) = e(M)$ . Using 1.2.4.b we get that

$$H(M, l-1) = \sum_{i=0}^{l-1} H(N, i) - b_{l-1}(M) = e(M)$$

Using 1.2.6.a we obtain  $\mathfrak{m}^l M = x\mathfrak{m}^{l-1} M$ . So we have that  $\mathfrak{m}^{i+1} M = x\mathfrak{m}^i M$  for all  $i \geq l-1$ . So we obtain that  $b_i(M) = 0$  for all  $i \geq l-1$ . This combined with 1. yields that  $x^*$  is  $G(M)$ -regular.  $\square$

An interesting consequence of the lemma above is the following lemma which gives information about the Hilbert function of a MCM module over a hypersurface ring of dimension 1.

**Lemma 4.7.** Let  $(Q, \mathfrak{n})$  be a regular local ring of dimension two,  $f \in \mathfrak{n}^e \setminus \mathfrak{n}^{e+1}$ ,  $e \geq 2$ ,  $A = Q/(f)$ . If  $M$  is a MCM  $A$ -module, then

$$h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M)z^i \quad \text{and } h_i(M) \geq 0 \ \forall i.$$

*Proof.* As  $\dim M = 1$ , the Hilbert series of  $M$  is  $h_M(z)/(1-z)$ . The Hilbert function of  $M$  is non-decreasing by Theorem 1. Therefore all the coefficients of  $h_M(z)$  are non-negative. Set  $b_i(M) = \lambda((\mathfrak{m}^{i+1}M :_M x)/\mathfrak{m}^i M)$ . Since

$$A^n \xrightarrow{\phi \otimes A} A^n \rightarrow M \rightarrow 0$$

is exact and all the entries of  $\phi$  are in  $i(M)$  we get by Lemma 4.6.1 that  $b_i(M) = 0$  for  $i = 0, \dots, i(M) - 1$ . This and Remark 4.1.1 yields that

$$h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M).$$

□

Next we get an upper bound on  $l$  such that  $\mathfrak{m}^l M \subseteq xM$  holds.

**Remark 4.8.** If  $\dim A = 1$  and  $x$  is  $A$ -superficial then note that since the ring  $B$  has length  $e_0(A)$  we get that  $\mathfrak{m}^{e_0(A)} \subseteq (x)$ . Therefore if  $\dim A = 1$ ,  $M$  a maximal  $A$ -module and  $x$  is  $A \oplus M$ -superficial then  $\mathfrak{m}^{e_0(A)} M \subseteq (x)M$

The next lemma deals with the case when  $M$  is a syzygy of a MCM  $A$ -module.

**Lemma 4.9.** *Let  $(A, \mathfrak{m})$  be a CM  $A$ -module of dimension 1 and let  $L$  be a non free MCM  $A$ -module. Set  $M = \text{Syz}_1^A(L)$ . If  $x$  is  $(A \oplus M \oplus L)$ -superficial then  $\mathfrak{m}^{e_0(A)-1} M \subseteq (x)M$ .*

*Proof.* By Remark 4.8 we have  $\mathfrak{m}^{e_0(A)} \subseteq (x)$ . We also have an exact sequence  $: 0 \rightarrow M \rightarrow F \rightarrow L \rightarrow 0$  where  $F$  is a free  $A$ -module. Set  $G = F/xF$  and  $W = L/xL$ . Going mod  $x$  we get  $0 \rightarrow N \rightarrow G \rightarrow W \rightarrow 0$ . Note that  $N \subseteq \mathfrak{n}G$ . Therefore  $\mathfrak{n}^{e_0(A)-1} N \subseteq \mathfrak{n}^{e_0(A)} G = 0$ . It follows that  $\mathfrak{m}^{e_0(A)-1} M \subseteq xM$ . □

*Proof of Theorem 2.* Clearly we may assume that  $k = Q/\mathfrak{n}$  is infinite. Let  $0 \rightarrow Q^n \xrightarrow{\phi} Q^n \rightarrow M \rightarrow 0$  be a minimal presentation of  $M$  over  $Q$ . If  $\dim A \geq 2$ , then choose  $x_1, \dots, x_d$  to be a maximal  $\phi$ -superficial sequence. Set  $J = (x_1, \dots, x_{d-1})$ . Since all the invariants considered in the theorem remain same modulo  $J$  it suffices to assume  $\dim A \leq 1$ . When  $\dim A = 0$  then all the results follow easily by Remark 4.2.

Therefore assume that  $\dim A = 1$ . Let  $x$  be  $\phi$ -superficial. Set  $R = Q/xQ$ ,  $N = M/xM$ ,  $\bar{f}$  = the image of  $f$  in  $R$  and  $B = A/xA = R/(\bar{f})$ . Note that

- a.  $i(M) = i(N)$ .
- b.  $e(M) = e(N)$  and  $e_1(M) \geq e_1(N)$  ( by 1.2.3 and 1.2.4 )
- c.  $R$  is a DVR with maximal ideal say  $\mathfrak{q} = (y)$ .
- d.  $v_R(f) = v_Q(f) = e$ .

So 1. follows from Remark 4.2 and (b) above.

2. If  $i(M) = e$  then since  $i(M) = i(N)$  we get by Remark 4.2.4 that  $N$  is a free  $B$  module. So  $M$  is a free  $A$ -module. Conversely if  $M$  is free then clearly  $i(M) = e$ .

3. Let  $M = F \oplus L$  where  $F$  is a free  $A$ -module and  $L$  has no free summands. Note that  $i(L) = i(M)$ . Since  $G(A)$  is CM it suffices to show  $G(L)$  is CM. Notice  $L = \text{Syz}_1^A(\text{Syz}_1^A(L))$ . If  $x$  is a  $A \oplus L \oplus N$ -superficial, element, then by Lemma 4.9 we get  $\mathfrak{m}^{e_0(A)-1} L \subseteq xL$ . Since  $A^n \rightarrow A^n \rightarrow L \rightarrow 0$  is exact and  $\text{depth } G(A) = 1$  we get by Lemma 4.6.2 that  $G(L)$  is CM.

4. By Proposition 4.7 we get that

$$h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(M)z^i \quad \text{and } h_i(M) \geq 0 \quad \forall i.$$

It follows that (i) and (ii) are equivalent. The assertion (iii)  $\implies$  (ii) is clear.

(i)  $\implies$  (iii). Note that  $\mu(N) = \mu(M)$  and

$$h_N(z) = \mu(M)(1 + z + \dots + z^{i(M)-1}) + \sum_{i \geq i(M)} h_i(N)z^i.$$

Also all the coefficients are non-negative. Therefore  $e(M) = e(N) = i(M)\mu(M)$  if and only if  $h_M(z) = h_N(z) = \mu(M)(1 + z + \dots + z^{i(M)-1})$ . Since  $h_M(z) = h_N(z)$  we also get that  $G(M)$  is CM (see 1.2.8).

Note that since  $M$  is not free  $i(M) < e$ . We first assert that  $M$  has no free summands. Otherwise  $M = F \oplus W$  where  $F$  is free. This yields  $h_M(z) = h_F(z) + h_W(z)$ . Since all the coefficients of  $h_F(z)$  and  $h_W(z)$  are non-negative we get that coefficient of  $z^{e-1}$  is non-zero. This contradicts (c). Therefore if  $(\phi, \psi)$  is a matrix-factorization of  $M$  then we have a minimal presentation of  $K$

$$0 \longrightarrow Q^n \xrightarrow{\psi} Q^n \longrightarrow K \longrightarrow 0.$$

Let  $x$  be both  $\phi$  and  $\psi$ -superficial. Set  $N = M/xM$  and  $R = Q/(x)$ . Since (a) holds then note that  $N \cong (R/(y^{i(M)}))^{\mu(M)}$ . Then

$$\text{Syz}_1^B(N) \cong (R/(y^{e-i(M)}))^{\mu(M)}.$$

Since  $\text{Syz}_1^B(N) \cong K/xK$  we get that  $i(K) = e - \mu(M)$  and so

$$e(K) = e(K/xK) = \mu(K/xK)i(K) = \mu(K)i(K).$$

Therefore by the equivalence of (i) and (iii) we get the required result.  $\square$

**Remark 4.10.** Theorem 2 can be applied to the case of Ulrich modules, that is, MCM modules that satisfy  $e(M) = \mu(M)$ . It is known, see [7], that Ulrich  $A$ -modules exist when  $A$  is a complete hypersurface ring. Using the previous theorem we get that if  $M$  is Ulrich, then  $i(M) = 1$  and so  $G(\text{Syz}_1^A(M))$  is CM. Furthermore  $i(\text{Syz}_1^A(A)) = e - 1$  and  $h_{\text{Syz}_1^A(M)} = \mu(M)(1 + z + \dots + z^{e-2})$ .

An easy way to test the hypothesis of the previous theorem in the equicharacteristic case is the following:

**Proposition 4.11.** *Let  $Q = k[[y_1, \dots, y_{d+1}]]$ . Let  $M$  be a  $Q$ -module with a minimal presentation  $0 \rightarrow Q^n \xrightarrow{\phi} Q^n \rightarrow M \rightarrow 0$ . Set*

$$\phi = \sum_{i \geq i(M)} \phi_i \quad \text{where } \phi_i \text{ are forms of degree } i$$

*Then  $\det \phi_{i(M)} \neq 0$  if and only if  $h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1})$ .*

*Proof.* Note that  $\det \phi_{i(M)} \neq 0$  if and only if  $v_Q(\det \phi) = i(M)\mu(M)$ .

Let  $f = \det \phi$ . Note that  $M$  is a maximal  $A = Q/(f)$ -module. Let  $x_1, \dots, x_d$  be a maximal  $\phi$ -superficial sequence. Set  $J = (x_1, \dots, x_d)$ ,  $R = Q/J$ ,  $\bar{f}$  = image of  $f$  in  $R$ ,  $N = M/JM$  and  $\bar{\phi} = \phi \otimes Q/J$ . Note that

$$i(N) = i(M) \quad \text{and} \quad v_R(\det \bar{\phi}) = v_Q(\det \phi).$$

If  $\det \phi_{i(M)} \neq 0$  then  $v_Q(\det \phi) = i(M)\mu(M)$ . So  $v_R(\det \bar{\phi}) = i(M)\mu(M)$ . Therefore by Remark 4.2.7 we get that  $e(N) = \mu(N)i(N)$ . This yields  $e(M) = i(M)\mu(M)$  and so by Theorem 2 we get the required assertion. Conversely if  $h_M(z) = \mu(M)(1 + z + \dots + z^{i(M)-1})$  then by Theorem 2 we get that  $G(M)$  is CM. So  $h_N(z) = h_M(z)$ . So we get  $e(N) = \mu(N)i(N)$ . Therefore by Remark 4.2.7 we get  $v_R(\det \bar{\phi}) = i(N)\mu(N) = i(M)\mu(M)$ . So  $v_Q(\det \phi) = i(M)\mu(M)$ .  $\square$

We give an application of the proposition proved above.

**Example 4.12.** Set  $Q = k[[y_1, \dots, y_{d+1}]]$  and  $\mathfrak{n}$  to be the maximal ideal of  $Q$ . Let  $a, b, c, d$  be in  $\mathfrak{n}$  be such that  $f = ad - bc \neq 0$ . Set  $A = Q/(f)$ . Set

$$\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Define  $M$  and  $K$  by the exact sequences

$$0 \rightarrow Q^2 \xrightarrow{\phi} Q^2 \rightarrow M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow Q^2 \xrightarrow{\psi} Q^2 \rightarrow K \rightarrow 0$$

Note that  $M$  and  $K$  are  $A$ -modules and  $K = \text{Syz}_1^A(M)$ .

1. If  $f \in \mathfrak{m}^2 \setminus \mathfrak{m}^3$  then  $M$  is an Ulrich  $A$ -module.
2. If  $f \in \mathfrak{m}^3 \setminus \mathfrak{m}^4$  then  $M$  or  $K$  has minimal multiplicity. Both are not Ulrich.

To prove 1. note that  $i(M) = 1$  and  $\det(\phi_1) \neq 0$ . So by Proposition 4.11 we get that  $h_M(z) = \mu(M)$ . So  $M$  is Ulrich.

2. We first show that  $M$  and  $K$  are not Ulrich. As  $f \in \mathfrak{n}^3 \setminus \mathfrak{n}^4$  we have  $i(M) = 1$ . Since  $f \in \mathfrak{n}^3$  we also get  $\det(\phi_1) = 0$ . By Proposition 4.11 we get that  $h_M(z) \neq \mu(M)$ . So  $M$  is not Ulrich. Similarly we get  $K$  is not Ulrich. Notice  $h_A(z) = 1 + z + z^2$ . By (4) and Proposition 1.5.(iv) we have

$$\mu(M)\chi_1(A) - \chi_1(M) - \chi_1(K) = \chi_0(l_M(z)) = l_M(1) - l_M(0).$$

Notice  $l_M(0) = \mu(K)$ . By [10, Lemma 19]  $l_M(1) \geq e_0(K)$ . Since  $K$  is not Ulrich we have  $\chi_0(l_M(z)) > 0$ . It follows that  $\chi_1(M) = 0$  or  $\chi_1(K) = 0$ . By 1.2.9 we get that  $M$  or  $K$  has minimal multiplicity.

## 5. SECOND HILBERT COEFFICIENT

**5.1.** In this section  $A$  is CM and  $M$  is a MCM  $A$ -module. Set  $k = A/\mathfrak{m}$ .

In view of 1 is natural to ask how do higher Hilbert coefficients of  $A$ ,  $M$  and  $\text{Syz}_1^A(M)$  are related. Lemma 2.8 indicates a way.

**Theorem 5.2.** (with hypothesis as in 5.1) Assume we have an exact sequence  $0 \rightarrow M \rightarrow F \rightarrow E \rightarrow 0$  with  $F$  free  $A$ -module and  $E$  a finite MCM  $A$ -module. If  $G(M)$  is CM and  $\text{depth } G(A) \geq d - 1$  then

1.  $e_2(F) \geq e_2(M) + e_2(E)$  and  $\chi_2(F) \geq \chi_2(M) + \chi_2(E)$ .
2.  $e_i(F) \geq e_i(M)$  and  $\chi_i(F) \geq \chi_i(M)$  for  $i \geq 0$ .

Theorem 5.2 is not satisfactory as there is no easy criteria for finding an MCM  $A$ -module  $E$  with  $G(\text{Syz}_1^A(E))$  is CM. However if  $A$  is Gorenstein then every MCM  $A$ -module is a syzygy of a MCM  $A$ -module. This can be seen as follows: Let  $M$  be a MCM  $A$ -module. The module  $M^* = \text{Hom}_A(M, A)$  is also a MCM  $A$ -module. Let  $N = \text{Syz}_1^A(M^*)$  and  $F = A^{\text{type}(M)}$ . We have an exact sequence  $0 \rightarrow N \rightarrow F \rightarrow M^* \rightarrow 0$ . Dualizing we get  $0 \rightarrow M^{**} \rightarrow F^* \rightarrow N^* \rightarrow 0$ . Note that  $M \cong M^{**}$ . Set  $S^A(M) = N^*$ . Interestingly  $S^A(M)$  behaves well mod superficial sequences.

**Lemma 5.3.** (with hypothesis as in 5.1). Let  $A$  be a Gorenstein ring. If  $x$  is  $A \oplus M$ -regular then for the  $B = A/(x)$ -module  $N = M/xM$  we have

1.  $\text{Hom}_B(N, B) \cong M^*/xM^*$ .
2.  $\text{Syz}_1^B(M^*/xM^*) \cong \text{Syz}_1^A(M^*)/x\text{Syz}_1^A(M^*)$ .
3.  $S^B(N) \cong S^A(M)/xS^A(M)$ .

*Proof.* 1. We use  $0 \rightarrow M \xrightarrow{x} M \rightarrow N \rightarrow 0$  to get a long exact sequence

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_A(N, A) \longrightarrow \operatorname{Hom}_A(M, A) \xrightarrow{x} \operatorname{Hom}_A(M, A) \\ \longrightarrow \operatorname{Ext}_A^1(N, A) \longrightarrow \operatorname{Ext}_A^1(M, A). \end{aligned}$$

Notice  $\operatorname{Ext}_A^1(M, A) = 0$  as  $M$  is MCM and  $A$  is Gorenstein [2, 3.3.10.d]. Using the isomorphisms  $\operatorname{Hom}_A(N, A) = 0$  and  $\operatorname{Ext}_A^1(N, A) \cong \operatorname{Hom}_B(N, B)$  (see [2, 3.1.16]), we get 1. Note that 2. holds since  $M^*$  is MCM. To prove 3. we use 1. to get

$$S^A(M)/xS^A(M) = \frac{\operatorname{Hom}_A(\operatorname{Syz}_1^A(M^*), A)}{x \operatorname{Hom}_A(\operatorname{Syz}_1^A(M^*), A)} \cong \operatorname{Hom}_B\left(\frac{\operatorname{Syz}_1^A(M^*)}{x \operatorname{Syz}_1^A(M^*)}, B\right)$$

$$S^B(N) = \operatorname{Hom}_B(\operatorname{Syz}_1^B(\operatorname{Hom}_B(N, B)), B) \cong \operatorname{Hom}_B(\operatorname{Syz}_1^B(M^*/xM^*), B)$$

Finally we use 2. to get the result.  $\square$

The following example shows that the hypothesis on  $\operatorname{depth} G(A)$  in Theorem 5.2 cannot be dropped.

**Example 5.4.** Set  $R = k[[x, y, z, u, v]]$  and  $\mathfrak{q} = (z^2, zu, zv, uv, yz - u^3, xz - v^3)$ . Set  $A = R/\mathfrak{q}$ ,  $E = R/\mathfrak{q} + (z)$  and  $M = (\mathfrak{q} + (z))/\mathfrak{q}$ . The ring  $A$  is CM and by [1] we get  $h_A(t) = 1 + 3t + 3t^3 - t^4$ . It is known  $\operatorname{depth} G(A) = 0$ , see [3, 3.10]. Note that  $M$  and  $E$  are  $A$ -modules and we have an obvious exact sequence  $0 \rightarrow M \rightarrow A \rightarrow E \rightarrow 0$ . We show

- (a)  $E$  and  $M$  are MCM  $A$ -modules and  $G(E)$  and  $G(M)$  are CM.
- (b)  $\mu(E)e_3(A) \not\geq e_3(M) + e_3(E)$  and  $\mu(E)e_3(A) \not\geq e_3(M)$ .

*Proof* (a). If we prove  $E$  is MCM then it follows that  $M$  is MCM. Notice

$$E = \frac{k[[x, y, u, v]]}{(z, uv, u^3, v^3)} \cong \frac{k[[x, y, u, v]]}{(uv, u^3, v^3)} \cong \frac{k[[u, v]]}{(uv, u^3, v^3)}[[x, y]].$$

So  $E$  is MCM. Clearly  $G_{\mathfrak{m}}(E) = \frac{k[[u, v]]}{(uv, u^3, v^3)}[x^*, y^*]$ . So  $G(E)$  is CM and  $h_E(t) = 1 + 2t + 2t^2$ . Notice  $M$  is a cyclic  $A$ -module. Since  $(z, u, v)M = 0$  we get that  $M$  is a  $S = A/(z, u, v) = k[[x, y]]$  module. Since  $M$  is also MCM  $S$ -module it is free. As  $M$  is cyclic and free  $M \cong S$ . Thus  $G(M) \cong k[x^*, y^*]$  is CM and  $h_M(t) = 1$ .

(b) Note that  $e_3(A) = -1$ ,  $\mu(E) = 1$ ,  $e_3(M) = e_3(E) = 0$ .

*Proof of Theorem 5.2.* We prove the result regarding  $e_i$ . The result regarding  $\chi_i$  can be proved on similar lines. By Remark 1.1 we may assume  $k$  is infinite.

When  $\dim A = 1$ , assertion 1. follows from Lemma 2.8. When  $\dim M = 2$ , let  $x$  be  $M \oplus E \oplus A$ -superficial. Set  $(B, \mathfrak{n}) = (A/(x), \mathfrak{m}/(x))$ ,  $N = M/xM$  and  $G = F/xF$  and  $L = E/xL$ . Note that  $B$  is one-dimensional CM ring and  $N, L$  are MCM  $B$  modules, and we have an exact sequence  $0 \rightarrow N \rightarrow G \rightarrow L \rightarrow 0$ .

Since  $G(M)$  is Cohen-Macaulay and  $\operatorname{depth} G(A) \geq 1$  we have that  $e_2(M) = e_2(N)$  and  $e_2(F) = e_2(G)$ . Furthermore it follows from 1.1.6 that  $e_2(E) \leq e_2(L)$ . Therefore we have:

$$e_2(M) + e_2(E) \leq e_2(N) + e_2(L) \leq e_2(G) = e_2(F).$$

Note that the second inequality above follows from the dimension one case.

When  $\dim A > 2$  let  $x_1, \dots, x_{d-2}$  be a  $M \oplus E \oplus A$ -superficial sequence. Set  $J = (x_1, \dots, x_{d-2})$ ,  $(B, \mathfrak{n}) = (A/J, \mathfrak{m}/J)$ ,  $N = M/JM$ ,  $G = F/JF$  and  $L = E/JE$ . By 1.1.6 we get that  $e_2(E) = e_2(L)$ ,  $e_2(M) = e_2(N)$  and  $e_2(F) = e_2(G)$ . So the result follows from the dimension 2 case.

To prove (ii) note that since  $G(M)$  is Cohen-Macaulay and  $\text{depth } G(A) \geq d-1$  it suffices to consider the case when  $\dim A = 1$ . By 2.8.4 we get that  $e_i(S^A(M)) \geq 0$  for all  $i \geq 1$ . So we get  $e_i(A)\mu(M) \geq e_i(M)$  by Lemma 2.8.  $\square$

Theorem 3 now follows as a corollary to Theorem 5.2.

*Proof of Theorem 3.* By Remark 1.1 we may assume  $k$  is infinite. Set  $L^* = S^A(M)$  and  $F^* = A^{\text{type}(M)}$ . We have an exact sequence  $0 \rightarrow M \rightarrow F^* \rightarrow L^* \rightarrow 0$ . The assertion of Theorem 3 follow from Theorem 5.2.  $\square$

## 6. FIRST HILBERT COEFFICIENT OF THE CANONICAL MODULE

In this section  $(A, \mathfrak{m})$  is a CM local ring with a canonical module  $\omega_A$ . It is well known that  $e_0(\omega_A) = e_0(A)$ . In Theorem 4 we obtain bounds for  $e_1(\omega_A)$ .

*Proof of Theorem 4.* Using [10, Theorem 18] it follows that  $e_1(\omega_A) \leq \tau e_1(A)$  with equality if and only if  $A$  is Gorenstein. We prove the lower bound on  $e_1(\omega_A)$ . There is nothing to prove when  $A$  is Gorenstein. So we assume  $A$  is *not* Gorenstein. By Remark 1.1 we may assume  $k = A/\mathfrak{m}$  is infinite.

**Reduction to dimension 1:** Let  $x_1, \dots, x_d$  be a maximal  $\omega_A \otimes A$  superficial sequence. Set  $J = (x_1, \dots, x_{d-1})$  and  $B = A/J$ . Clearly  $B$  is a CM local ring of dimension 1. We also have  $\omega_B \cong \omega_A/J\omega_A$ ; cf. [2, Theorem 3.3.5]. By Remark 1.2(3) we have  $e_1(B) = e_1(A)$  and  $e_1(\omega_B) = e_1(\omega_A)$ . Also  $\text{type}(A) = \text{type}(B)$ .

(1.) Set  $\omega = \omega_A$  and  $M^\dagger = \text{Hom}_A(M, \omega)$ . We dualize the exact sequence  $0 \rightarrow N \rightarrow A^\tau \rightarrow \omega \rightarrow 0$ , to obtain the exact sequence (see [2, Theorem 3.3.10(d)])

$$(*) \quad 0 \longrightarrow A \longrightarrow \omega^\tau \longrightarrow N^\dagger \longrightarrow 0.$$

Let  $x$  be  $A \oplus \omega \oplus N^\dagger$ -superficial. For  $i \geq 0$  give  $L_i(N^\dagger)$ ,  $L_i(\omega)$ , and  $L_i(A)$  the  $A[X]$ -module structure as described in Remark 2.1.

By Proposition 2.2.1 we have an exact sequence

$$(a) \quad L_1(N^\dagger) \xrightarrow{\delta} L_0(A) \rightarrow L_0(\omega^\tau) \rightarrow L_0(N^\dagger) \rightarrow 0. \quad \text{Set } K = \text{image } \delta.$$

Note that  $N^\dagger$  is not free, since otherwise by (\*) we get  $\omega$  is free, a contradiction; since  $A$  is *not* Gorenstein. Using Lemma 2.8 there exists an  $\mathfrak{m}$ -primary ideal  $\mathfrak{q}$  such that  $L_1(N^\dagger)$  is a finitely generated  $R = A/\mathfrak{q}[X]$ -module of dimension 1. So  $K = \bigoplus_{n \geq 0} K_n$  is a finitely generated  $R$ -module. Furthermore  $\dim K \leq \dim R = 1$ . Set  $\sum_{n \geq 0} \lambda(K_n)z^n = l_K(z)/(1-z)$  and  $l_K(1) \geq 0$ . Note that  $l_K(1) \neq 0$  iff  $\dim K = 1$ . Using (a) we get

$$(b) \quad (1-z)l_K(z) = -\tau h_\omega(z) + h_A(z) + h_{N^\dagger}(z) \quad \text{and so}$$

$$(c) \quad \tau e_1(\omega) = e_1(A) + e_1(N^\dagger) + l_K(1).$$

Since all the terms involved in (c) are non-negative it follows that

$$(d) \quad \tau e_1(\omega) \geq e_1(A) \quad \text{with equality iff } e_1(N^\dagger) = l_K(1) = 0.$$

This proves the assertion for lower bound of  $e_1(\omega_A)$  in (1).

(3.) If  $G(A)$  is CM then  $x^*$  is  $G(A)$ -regular (1.2.5.a). Using Proposition 2.2.3 we get that  $X$  is  $L_0(A)$ -regular. From (a) it follows that either  $K = 0$  or  $X$  is an  $K$ -regular element.

If  $K = 0$  then using (b) we get that  $\tau h_\omega(z) = h_A(z) + h_{N^\dagger}(z)$ . So  $\tau e_i(\omega_A) = e_i(A) + e_i(N^\dagger)$  for all  $i \geq 0$ . By (1.2.6.b)  $e_i(N^\dagger) \geq 0$  for all  $i \geq 0$ . Thus  $\tau e_i(\omega_A) \geq e_i(A)$  for all  $i \geq 0$ .

If  $K \neq 0$  then  $X$  is  $K$ -regular. Also  $\dim K = 1$ . Thus  $K$  is a CM  $R$ -module. So all the coefficients of  $l_K(z)$  are non-negative. Using (b) we get

$$\tau e_i(\omega_A) - e_i(A) - e_i(N^\dagger) = l_K^{(i-1)}(1)/(i-1)! \geq 0$$

for  $i \geq 1$ . This gives (3), since  $\dim N^\dagger = 1$  and so  $e_i(N^\dagger) \geq 0$  for each  $i \geq 1$ .

(2.) Set  $k = A/\mathfrak{m}$ . If  $e_1(A) = \tau e_1(\omega)$  then from (d) it follows that  $e_1(N^\dagger) = 0$ . Therefore  $N^\dagger$  (and so  $N \cong (N^\dagger)^\dagger$ ) are Ulrich  $A$ -modules. So  $\mu(N) = e_0(N) = (\tau - 1)e_0(A)$ . Set  $e = e_0(A)$ . Let  $y$  be a  $N \oplus \omega \oplus A$ -superficial element. Set  $(B, \mathfrak{n}) = (A/(y), \mathfrak{m}/(y))$ . Note that  $N/xN = k^{(\tau-1)e}$  and  $\omega_B = \omega/y\omega$ . We also have an exact sequence  $0 \rightarrow k^{(\tau-1)e} \rightarrow B^\tau \rightarrow \omega_B \rightarrow 0$ . This yields an exact sequence

$$(f) \quad 0 \longrightarrow \text{Hom}_B(k, k^{(\tau-1)e}) \longrightarrow \text{Hom}_B(k, B^\tau) \longrightarrow \text{Hom}_B(k, \omega_B) \longrightarrow \cdots$$

Note that  $\text{Hom}_B(k, B^\tau) \cong k^{\tau^2}$  and  $\text{Hom}_B(k, \omega_B) \cong k$ . So by (f) we get that

$$\text{either } \tau^2 = (\tau - 1)e + 1 \quad \text{or} \quad \tau^2 = (\tau - 1)e.$$

Case 1:  $\tau^2 = (\tau - 1)e + 1$ .

Since  $\tau \neq 1$ , we get  $\tau = e - 1$ . So  $\tau = \lambda(\mathfrak{n})$ . Thus  $\text{socle}(B) = \mathfrak{n}$ . Therefore  $\mathfrak{n}^2 = 0$ . So  $B$  has minimal multiplicity and therefore  $A$  also has minimal multiplicity.

Case 2:  $\tau^2 = (\tau - 1)e$ .

So  $\tau^2 - 1 = (\tau - 1)e - 1$ . As  $\tau \neq 1$ , we get  $\tau + 1 = e - [1/(\tau - 1)]$ . Therefore  $\tau \not\geq 3$ . As  $A$  is not Gorenstein we have  $\tau = 2$  and  $\mu(\mathfrak{n}) \geq 2$ . So  $e = 4$ . There exists two possible Hilbert series for  $B$ , namely

$$(i) \ h_B(z) = 1 + 3z \quad \text{or} \quad (ii) \ h_B(z) = 1 + 2z + z^2.$$

Claim: (ii) is not possible. *Proof:* Note that  $h = \mu(\mathfrak{m}) - 1 = \mu(\mathfrak{n}) = 2$ . So  $e = h + 2$  and  $h = 2 = \tau$ . Then by [13, Theorem 6.12], we get  $h_A(z) = 1 + 2z + z^3$ . So  $e_1(A) = 5 \neq 2e_1(\omega)$ , a contradiction. Thus only (i) holds and so  $B$  (and therefore  $A$ ) has minimal multiplicity.

Conversely if  $A$  has minimal multiplicity then  $\omega$  also has minimal multiplicity. In particular  $G(A)$  and  $G(\omega)$  are both CM. Say  $h_A(z) = 1 + hz$ . It can be checked that  $h = \tau$ . It follows that  $h_\omega(z) = h + a_1z$ . Since  $e_0(A) = e_0(\omega)$  we get  $a_1 = 1$ . Therefore we get  $e_1(A) = \tau e_1(\omega)$ .  $\square$

## 7. GENERIC

**7.1.** In this section  $A = k[[x_1, \dots, x_\nu]]/\mathfrak{q}$  is CM of dimension  $d \geq 1$  and  $\mathfrak{q} \subseteq (\mathbf{x})^2$ . Unless stated otherwise the field  $k$  is assumed to be infinite.

**7.2.** Let  $1 \leq r \leq d$ . Set  $y_j = \sum_{i=1}^r \alpha_{ij} x_i$  for  $j = 1, \dots, r$  and  $\alpha_{ij} \in k$ . We prove that for 'sufficiently general (= s.g.)'  $\alpha_{ij}$ , the Hilbert function of  $A/(\mathbf{y})$  remains the same. Notice for s.g.  $\alpha_{ij}$  we get  $\mathbf{y}$  to be a superficial sequence.

**Construction 7.3.** We describe a construction due to Marley [8, p. 32]. Let  $y_1, \dots, y_r$  be an  $A$ -superficial sequence. Let  $\mathbf{K}_\bullet(\mathbf{y})$  be the Koszul complex and for each  $n \geq 1$  let  $\mathbf{K}_\bullet^{(n)}(\mathbf{y})$  be the subcomplex

$$0 \rightarrow \mathfrak{m}^{n+1-r} K_r \rightarrow \cdots \rightarrow \mathfrak{m}^n K_1 \rightarrow \mathfrak{m}^{n+1} K_0 \rightarrow 0$$

Let  $\mathbf{C}_\bullet^{(n)}(\mathbf{y}) = \mathbf{K}_\bullet(\mathbf{y})/\mathbf{K}_\bullet^{(n)}(\mathbf{y})$ . So  $\mathbf{C}_\bullet^{(n)} = \mathbf{C}_\bullet^{(n)}(\mathbf{y})$  is

$$0 \rightarrow \frac{A}{\mathfrak{m}^{n+1-r}} \xrightarrow{\psi_{n,r}^{\mathbf{y}}} \dots \rightarrow \left(\frac{A}{\mathfrak{m}^{n-1}}\right)^{\binom{r}{2}} \xrightarrow{\psi_{n,2}^{\mathbf{y}}} \left(\frac{A}{\mathfrak{m}^n}\right)^{\binom{r}{1}} \xrightarrow{\psi_{n,1}^{\mathbf{y}}} \frac{A}{\mathfrak{m}^{n+1}} \rightarrow 0.$$

Clearly  $H_0(\mathbf{C}_\bullet^{(n)}) = A/(\mathbf{y}, \mathfrak{m}^{n+1})$ . Also  $H_i(\mathbf{C}_\bullet^{(n)}) = 0$  for  $i \geq 1$  and  $n \gg 0$  [6, 3.6].

**Remark 7.4.** As  $\mathbf{C}_\bullet^{(n)}(\mathbf{y})$  is a bounded complex of modules of finite length we get

$$(7) \quad \sum_{i=0}^r (-1)^i \lambda \left( \mathbf{C}_\bullet^{(n)}(\mathbf{y}) \right) = \sum_{i=0}^r (-1)^i \lambda \left( H_i(\mathbf{C}_\bullet^{(n)}(\mathbf{y})) \right).$$

If  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  is a function, set  $\Delta(f) = f(n) - f(n-1)$ . Note (7) yields

$$\begin{aligned} \Delta^r \lambda \left( \frac{A}{\mathfrak{m}^{n+1}} \right) &= \lambda \left( \frac{A}{(\mathbf{y}, \mathfrak{m}^{n+1})} \right) + w(\mathbf{y}, n); \text{ where} \\ w(\mathbf{y}, n) &= \sum_{i=1}^r (-1)^i \lambda \left( H_i(\mathbf{C}_\bullet^{(n)}(\mathbf{y})) \right). \end{aligned}$$

It follows from Marley's result that  $w(\mathbf{y}, n) = 0$  for  $n \gg 0$ . An easier way to see this is by using the Hilbert-Samuel polynomial. It can be easily checked that

$$(8) \quad w(\mathbf{y}, n) = 0 \quad \text{for } n \geq \max\{\text{post}(A) + r, \text{post}(A/(\mathbf{y}))\}.$$

Recall  $\text{post}(A)$  is the postulation number of the Hilbert-Samuel function, see (2).

**7.5.** The invariant  $\mathbf{m}(A)$  by Trivedi [12] is convenient for our purposes. Trivedi [12, Theorem 2] proved that  $\text{post}(A) \leq \mathbf{m}(A)$  when  $\dim A \geq 0$ . Also it can be easily checked that if  $y_1, \dots, y_r$  is a superficial sequence then  $\mathbf{m}(A/\mathbf{y}) \leq \mathbf{m}(A)$ .

Thus using (8) it follows that

$$(9) \quad w(\mathbf{y}, n) = 0 \quad \text{for } n \geq \mathbf{m}(A) + r.$$

**Lemma 7.6.** (with hypothesis as in 7.1) Let  $1 \leq r \leq d$ . Then for any two sets of  $r$ , s.g  $k$ -linear combinations of  $x_1, \dots, x_\nu$  say  $y_1, \dots, y_r$  and  $z_1, \dots, z_r$

$$H(A/(\mathbf{y}), n) = H(A/(\mathbf{z}), n) \quad \text{for each } n \geq 0.$$

*Proof.* We use 7.2, 7.3. We prove  $w(\mathbf{y}, n)$  is constant for s.g  $\alpha_{ij}$ . Since  $w(\mathbf{y}, n) = 0$  for all  $n \geq \mathbf{m}(A)$  for any superficial sequence, it suffices to show for each  $s \geq 1$  we have  $\dim_k H_s(\mathbf{C}_\bullet^{(n)}(\mathbf{y}))$  is constant for s.g  $\alpha_{ij}$ . Fix an integer  $s$  with  $1 \leq s \leq r$ .

$$\text{Consider } \psi_{n,s}^{\mathbf{y}}: \left(\frac{A}{\mathfrak{m}^{n+1-s}}\right)^{\binom{r}{s}} \rightarrow \left(\frac{A}{\mathfrak{m}^{n+2-s}}\right)^{\binom{r}{s-1}}.$$

Let  $\eta_{s-1} = \sup \{ \dim_k \text{image } \psi_{n,s}^{\mathbf{y}} \mid \mathbf{y} = y_1, \dots, y_r \text{ is a superficial sequence} \}.$

$$\text{Set } y_j = \sum_{i=1}^{\nu} \alpha_{ij} x_i, \quad \alpha_{ij} \in k.$$

**Claim:** For s.g  $\alpha_{ij}$   $\dim_k \text{image } \psi_{n,s}^{\mathbf{y}} = \eta_{s-1}.$

**7.7.** If we prove the claim then we are done since

$$\begin{aligned} \dim_k H_s(\mathbf{C}_\bullet^{(n)}(\mathbf{y})) &= \dim_k \ker \psi_{n,s}^{\mathbf{y}} - \dim_k \text{image } \psi_{n,s+1}^{\mathbf{y}} \\ &= \dim_k \left( \frac{A}{\mathfrak{m}^{n+1-s}} \right)^{\binom{r}{s}} - \dim_k \text{image } \psi_{n,s}^{\mathbf{y}} - \dim_k \text{image } \psi_{n,s+1}^{\mathbf{y}} \\ &= \dim_k \left( \frac{A}{\mathfrak{m}^{n+1-s}} \right)^{\binom{r}{s}} - \eta_{s-1} - \eta_s, \quad \text{for s.g. } \alpha_{ij}. \end{aligned}$$

*Proof of Claim:* Let  $\{u_1, \dots, u_g\}$  be a  $k$ -basis of  $A/\mathfrak{m}^{n+1-s}$  and let  $\{v_1, \dots, v_l\}$  be a  $k$ -basis of  $A/\mathfrak{m}^{n+2-s}$ . Then  $u_t e_{j_1} \wedge \dots \wedge e_{j_m}$  with  $1 \leq t \leq g$ , and  $1 \leq j_1 \leq \dots \leq j_s \leq r$  is a  $k$ -basis of  $(A/\mathfrak{m}^{n+1-s})^{\binom{r}{s}}$  and  $v_t e_{j_1} \wedge \dots \wedge e_{j_{s-1}}$  with  $1 \leq t \leq l$ , and  $1 \leq j_1 \leq \dots \leq j_{m-1} \leq r$  is a  $k$ -basis of  $(A/\mathfrak{m}^{n+2-s})^{\binom{r}{s-1}}$ . Set

$$u_t x_i = \sum_{\xi=1}^l a_{\xi}^{(t,i)} v_{\xi} \quad \text{for } t = 1, \dots, g \text{ and } i = 1, \dots, \nu.$$

Since  $y_j = \sum_{i=1}^{\nu} \alpha_{ij} x_i$ , we have for  $t = 1, \dots, g$  and  $j = 1, \dots, r$ ,

$$u_t y_j = \sum_{i=1}^{\nu} \alpha_{ij} \left( \sum_{\xi=1}^l a_{\xi}^{(t,i)} v_{\xi} \right) = \sum_{\xi=1}^l f_{\xi}^{(t,j)}(\underline{\alpha}) v_{\xi} \quad \text{where } f_{\xi}^{(t,j)}(\underline{\alpha}) = \sum_{i=1}^{\nu} \alpha_{ij} a_{\xi}^{(t,i)}.$$

$$\begin{aligned} \text{Then } \psi_{n,s}^{\mathbf{y}}(u_t e_{j_1} \wedge \dots \wedge e_{j_s}) &= \sum_{q=1}^s (-1)^{q+1} u_t y_{j_q} e_{j_1} \wedge \dots \wedge \hat{e}_{j_q} \wedge \dots \wedge e_{j_s} \\ &= \sum_{q=1}^s (-1)^{q+1} \left( \sum_{\xi=1}^l f_{\xi}^{(t,j_q)}(\underline{\alpha}) v_{\xi} \right) e_{j_1} \wedge \dots \wedge \hat{e}_{j_q} \wedge \dots \wedge e_{j_s} \\ &= \sum_{q=1}^s \sum_{\xi=1}^l (-1)^{q+1} f_{\xi}^{(t,j_q)}(\underline{\alpha}) v_{\xi} e_{j_1} \wedge \dots \wedge \hat{e}_{j_q} \wedge \dots \wedge e_{j_s} \end{aligned}$$

Thus the map  $\psi_{n,s}^{\mathbf{y}}$  can be described by a matrix of linear forms in  $\alpha_{ij}$ . Replacing the  $\alpha_{ij}$  by variables we get a matrix  $E$  of linear polynomials with coefficient in  $k$ . Then by construction  $U = k^{\nu} \setminus V(I_{\eta}(E))$  is non-empty open subset of  $k^{\nu}$ . For  $\alpha_{ij} \in U$ , we get  $\dim_k \text{image } \psi_{n,s}^{\mathbf{y}} = \eta_{s-1}$ . As indicated in 7.7 this finishes the proof.  $\square$

**Application:** We now give an application of Lemma 7.6 to bound Hilbert coefficients of a MCM module  $M$  if  $G(M)$  is CM.

**Theorem 7.8.** *Let  $(A, \mathfrak{m})$  be a equicharacteristic CM local ring of dimension  $d > 0$ . Then there exists a Artinian local ring  $R$  and a one dimensional Gorenstien local ring  $T$  with the property with the property*

(\*) *If  $M$  is any MCM  $A$ -module with  $G(M)$  CM then*

1.  $e_i(M) \leq e_i(R)\mu(M)$  and  $\chi_i(M) \leq \chi_i(R)\mu(M)$  for all  $i \geq 0$
2.  $e_i(M) \leq e_i(T)\text{type}(M)$  and  $\chi_i(M) \leq \chi_i(T)\text{type}(M)$  for all  $i \geq 0$ .

Furthermore if  $A = S/\mathfrak{q}$  where  $S = k[[x_1, \dots, x_n]]$  with  $k$  is infinite and  $\mathfrak{q} \subseteq (\mathbf{x})^2$  then one can take  $R = S/\mathfrak{q} + (v_1, \dots, v_d)$  and  $T = S/I + (v_1, \dots, v_{d-1})$  where the  $v_i$ 's

are s.g linear combination of  $X_1, \dots, X_n$  and  $I = (f_1, \dots, f_r)$  is an ideal contained in  $\mathfrak{q}$  with  $\text{grade } \mathfrak{q} = r$  and  $f_1, \dots, f_r$  is a regular sequence.

*Proof.* Using Remark 1.1 we may assume that the residue field  $k$  of  $A$  is infinite. Also we may assume  $A$  is complete. This we do. By Cohen structure theorem we can assume  $A = S/\mathfrak{q}$  where  $S = k[[x_1, \dots, x_n]]$  and  $\mathfrak{q} \subseteq (\mathfrak{x})^2$ .

1. Let  $R = A/(v_1, \dots, v_d) = S/\mathfrak{q} + (v_1, \dots, v_d)$ , where the  $v_i$ 's are s.g linear combination of  $x_1, \dots, x_n$ .

We know that if  $\mathbf{y} = y_1, \dots, y_d$  are s.g linear combination of  $x_1, \dots, x_n$ , then  $y_1, \dots, y_d$  is an  $M$ -superficial sequence and the Hilbert function of  $B = A/\mathbf{y}A$  is equal to Hilbert function of  $R$ , by Lemma 7.6.

Set  $N = M/\mathbf{y}M$ . By Remark 2.6 we have  $e_i(N) \leq e_i(B)\mu(N)$  for each  $i \geq 0$ . Note that  $\mu(N) = \mu(M)$  and since  $G(M)$  is CM  $e_i(M) = e_i(N)$ , for  $i \geq 0$ . Since the Hilbert function of  $B$  is same as that of  $R$  we have  $e_i(B) = e_i(R)$  for each  $i \geq 0$ . So we get the result.

2. First note that  $M$  is an MCM  $Q = S/I$ -module and  $Q$  is Gorenstein. Then one uses Theorem 3 and proves this assertion along the same lines as 1.  $\square$

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